Yet Another Proof of Sylvester's Identity

Paul Vrbik[1]

[1]University of Newcastle Australia

Sylvester's Identity

Let $\mathbb{A}$ be an $n \times n$ matrix with entries $a_{i,j}$ for $i, j \in [1, n]$ and denote by $\mathbb{A}_{i|k}$ the matrix $\mathbb{A}$ with Row i and Column k deleted; let $\mathbb{A}_{i,j|k,\ell}$ be the matrix obtained by deleting Rows i and j *and* Columns k and ℓ:

$$\mathbb{A}_{i|k} := \left[\begin{array}{ccc} & a_{k,1} & \\ a_{i,1} & & a_{i,n} \\ & a_{k,n} & \end{array}\right], \quad \mathbb{A}_{i,j|k,\ell} := \left[\begin{array}{cccc} & a_{k,1} & a_{\ell,1} & \\ a_{i,1} & & & a_{i,n} \\ a_{j,1} & & & a_{j,n} \\ & a_{k,n} & a_{\ell,n} & \end{array}\right].$$

Theorem (Sylvester's Determinant Identity)

For $\mathbb{A}$ a square matrix,

$$|\mathbb{A}| \cdot \left|\mathbb{A}_{i,j|k,\ell}\right| = \left|\mathbb{A}_{i|k}\right| \cdot \left|\mathbb{A}_{j|\ell}\right| - \left|\mathbb{A}_{i|\ell}\right| \cdot \left|\mathbb{A}_{j|k}\right|. \tag{1}$$

ELSEVIER

Mathematics and Computers in Simulation 42 (1996) 585–593

MATHEMATICS AND COMPUTERS IN SIMULATION

Various proofs of Sylvester's (determinant) identity

Alkiviadis G. Akritas [a,*], Evgenia K. Akritas [a], Genadii I. Malaschonok [b]

[a] *University of Kansas, Department of Computer Science, Lawrence, KS 66045-2192, USA*

[b] *Kiev University, Department of Cybernetics, Vladimirska 64, U-252017 Kiev, Ukraine*

Abstract

Despite the fact that the importance of Sylvester's determinant identity has been recognized in the past, we were able to find only one proof of it in English (Bareiss, 1968), with reference to some others. (Recall that Sylvester (1857) stated this theorem without proof.) Having used this identity, recently, in the validity proof of our new, improved, matrix-triangularization subresultant polynomial remainder sequence method (Akritas et al., 1995), we decided to collect all the proofs we found of this identity – one in English, four in German and two in Russian, in that order – in a single paper (Akritas et al., 1992). It turns out that the proof in English is identical to an earlier one in German. Due to space limitations two proofs are omitted.

The Seven Proofs 1/7

Bareiss's Proof

Based on the fact that

$$\mathbb{A}^{-1} = \frac{1}{|\mathbb{A}|} \cdot (\operatorname{adj} \mathbb{A})$$

where

$$(\operatorname{adj} \mathbb{A})_{i,k} = (-1)^{i+k} \, |\mathbb{A}|_{k|i}$$

and

$$(\operatorname{adj} \mathbb{A}) \cdot \mathbb{A} = |\mathbb{A}| \cdot$$

The Seven Proofs 2/7

Stuička

A straightforward (and rather long) induction proof. Understood "even by high school student."

The Seven Proofs 3/7

Kowalewski

An extended version of Bareiss. The only difference between them is that the extended version you do row operations rather than matrix multiplication.

The Seven Proofs 4/7

Kowalewski

An "elegant" proof based on Jacobi's identity.

Theorem (Jacobi's Identity)

Let $\mathbb{A} \neq 0$ *be a nonvanishing determinant and let* $|\operatorname{adj}\mathbb{A}|$ *be its adjoint determinant. Further let* $|(\operatorname{adj}\tilde{\mathbb{A}})^p|$ *be a minor of* $|\operatorname{adj}\mathbb{A}|$ *and let* $\tilde{a}^p_{rs}$ *be the corresponding minor of* $|\mathbb{A}|$*.*

Then $|(\operatorname{adj}\tilde{\mathbb{A}})^p|$ *differs from the algebraic complement of* $\tilde{a}^p_{rs}$ *by the factor* $|\mathbb{A}|^{p-1}$*.*

The Seven Proofs 5/7

Kowalewski

Proves the generalized identity.

The Seven Proofs 6/7 and 7/7

Malaschonok's

Malaschonok's proof differs from the others in that it does *not* require $\mathbb{A}$ to be given over a field and applies to matrices over any commutative ring. He gives two proofs predicated on the same two observations.

Theorem (Sylvester's Determinant Identity)

For $\mathbb{A}$ a square matrix,

$$|\mathbb{A}| \cdot \left|\mathbb{A}_{i,j|k,\ell}\right| = \left|\mathbb{A}_{i|k}\right| \cdot \left|\mathbb{A}_{j|\ell}\right| - \left|\mathbb{A}_{i|\ell}\right| \cdot \left|\mathbb{A}_{j|k}\right|.$$

Proof

We proceed by induction. As the sign of the determinant flips for each row or column that is permuted, without loss of generality, it is sufficient to show

$$|\mathbb{A}| \cdot \left|\mathbb{A}_{1,2|1,2}\right| = \left|\mathbb{A}_{1|1}\right| \cdot \left|\mathbb{A}_{2|2}\right| - \left|\mathbb{A}_{1|2}\right| \cdot \left|\mathbb{A}_{2|1}\right|. \qquad (2)$$

This simplifies the presentation somewhat.

Base

Recalling $|\mathbb{A}| = 1$ when $\mathbb{A}$ is 0×0 (is this *controversial* ?) it is easy to verify the result explicitly when $n = 2$

$$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} \cdot 1 = a_{2,2} \cdot a_{1,1} - a_{2,1} \cdot a_{1,2} = \left|\mathbb{A}_{1|1}\right| \cdot \left|\mathbb{A}_{2|2}\right| - \left|\mathbb{A}_{1|2}\right| \cdot \left|\mathbb{A}_{2|1}\right|.$$

Induction

We need some way to "extend" our matrices for the induction.

Notation ($\mathbb{A}^+$)

Extend $\mathbb{A}$ by an extra row and column and denote this new $(n+1) \times (n+1)$ matrix by $\mathbb{A}^+$.

Notation ($\mathbb{A}^{(i)}$)

Let $\mathbb{A}^{(i)}$ be the matrix obtained by replacing the i^{th} row of $\mathbb{A}$ with the last row of $\mathbb{A}^+$ (the row that was added) less the 'corner' element $a_{n+1,n+1}$.

Induction

Assuming the *induction hypothesis* premise

$$\left|\mathbb{A}\right| \cdot \left|\mathbb{A}_{i,j|k,\ell}\right| = \left|\mathbb{A}_{i|k}\right| \cdot \left|\mathbb{A}_{j|\ell}\right| - \left|\mathbb{A}_{i|\ell}\right| \cdot \left|\mathbb{A}_{j|k}\right|.$$

we need to prove

$$\left|\mathbb{A}^{+}\right| \cdot \left|\mathbb{A}^{+}_{1,2|1,2}\right| = \left|\mathbb{A}^{+}_{1|1}\right| \cdot \left|\mathbb{A}^{+}_{2|2}\right| - \left|\mathbb{A}^{+}_{1|2}\right| \cdot \left|\mathbb{A}^{+}_{2|1}\right|.$$

$|\mathbb{A}|$ *is linear in each* $a_{i,j}$ (it is a polynomial in n^2 variables with $n!$ terms comprised of a product of n distinct $a_{i,j}$'s).

Therefore the $a_{i,j}$ cofactor can be written as $\frac{\partial}{\partial a_{i,j}} |\mathbb{A}|$; expanding the determinant along the (say) j^{th} column:

$$|\mathbb{A}| = \sum_{i=1}^{n} a_{i,j} \cdot (-1)^{i+j} \left|\mathbb{A}_{i|j}\right| = \sum_{i=1}^{n} a_{i,j} \cdot \frac{\partial |\mathbb{A}|}{\partial a_{i,j}}.$$

Alternatively, using the i^{th} row gives

$$|\mathbb{A}| = \sum_{j=1}^{n} a_{i,j} \cdot \frac{\partial |\mathbb{A}|}{\partial a_{i,j}} \implies \left|\mathbb{A}^{(i)}\right| = \sum_{j=1}^{n} a_{n+1,j} \cdot \frac{\partial |\mathbb{A}|}{\partial a_{i,j}}$$

$\left|\mathbb{A}^{(i)}\right| = \sum_{j=1}^{n} a_{n+1,j} \cdot \frac{\partial}{\partial a_{i,j}} \left|\mathbb{A}\right|$ (the last equivalence) defines a linear differential operator $D^{(i)}$ with constant coefficients which obeys the product rule and commutes with $D^{(\ell)}$:

$$D^{(i)} := \sum_{j=1}^{n} a_{n+1,j} \cdot \frac{\partial}{\partial a_{i,j}}$$

This implies

$$\left|\mathbb{A}^{(i)}\right| = D^{(i)} \left|\mathbb{A}\right| .$$

Moreover $D^{(\ell)} \left|\mathbb{A}^{(i)}\right| = \mathit{0}$ because:

when $\ell \neq i$

The LHS results in a determinant of a matrix having two identical rows (the ℓ^{th} and j^{th}).

when $\ell = i$

$D^{(i)}$ is differentiating with respect to $a_{i,j}$ elements which are no longer part of the $D^{(i)} \left|\mathbb{A}\right|$ polynomial.

Induction

Expanding the determinant of $\mathbb{A}^+$ along its last column, we get

$$\left|\mathbb{A}^+\right| = a_{n+1,n+1}\left|\mathbb{A}\right| - \sum_{i=1}^{n} a_{i,n+1} D^{(i)}\left|\mathbb{A}\right| \tag{3}$$

since $-D^{(i)}\left|\mathbb{A}\right|$ is now the cofactor of $a_{i,n+1}$.

Using this equality to expand the determinants of our induction produces (left-hand-side)

$$\left(a_{n+1,n+1}\left|\mathbb{A}\right|-\sum_{i=1}^{n}a_{i,n+1}D^{(i)}\left|\mathbb{A}\right|\right)\cdot\left(a_{n+1,n+1}\left|\mathbb{A}_{1,2|1,2}\right|-\sum_{i=1}^{n}a_{i,n+1}D^{(i)}\left|\mathbb{A}_{1,2|1,2}\right|\right)$$

and (right-hand-side)

$$\left(a_{n+1,n+1}\left|\mathbb{A}_{1|1}\right|-\sum_{i=1}^{n}a_{i,n+1}D^{(i)}\left|\mathbb{A}_{1|1}\right|\right)\cdot\left(a_{n+1,n+1}\left|\mathbb{A}_{2|2}\right|-\sum_{i=1}^{n}a_{i,n+1}D^{(i)}\left|\mathbb{A}_{2|2}\right|\right)$$

$$-$$

$$\left(a_{n+1,n+1}\left|\mathbb{A}_{1|2}\right|-\sum_{i=1}^{n}a_{i,n+1}D^{(i)}\left|\mathbb{A}_{1|2}\right|\right)\cdot\left(a_{n+1,n+1}\left|\mathbb{A}_{2|1}\right|-\sum_{i=1}^{n}a_{i,n+1}D^{(i)}\left|\mathbb{A}_{2|1}\right|\right)$$

We expand this identity and *collect terms which have an* $n+1$ (the added terms):

1. $a^2_{n+1,n+1}$,
2. $a_{i,n+1}a_{j,n+1}$, and
3. $a_{n+1,n+1}a_{i,n+1}$.

Induction

Collecting *terms proportional to* $a_{n+1,n+1}^2$ produces the induction hypothesis.

Induction

Collecting *terms proportional to* $a_{i,n+1}a_{j,n+1}$ results in

$$\begin{aligned} & D^{(i)}\left|\mathbb{A}\right| \cdot D^{(j)}\left|\mathbb{A}_{1,2|1,2}\right| + D^{(j)}\left|\mathbb{A}\right| \cdot D^{(i)}\left|\mathbb{A}_{1,2|1,2}\right| \\ & \quad = D^{(i)}\left|\mathbb{A}_{1|1}\right| \cdot D^{(j)}\left|\mathbb{A}_{2|2}\right| + D^{(j)}\left|\mathbb{A}_{1|1}\right| \cdot D^{(i)}\left|\mathbb{A}_{2|2}\right| \\ & \qquad - D^{(i)}\left|\mathbb{A}_{1|2}\right| \cdot D^{(j)}\left|\mathbb{A}_{2|1}\right| - D^{(j)}\left|\mathbb{A}_{1|2}\right| \cdot D^{(i)}\left|\mathbb{A}_{2|1}\right| \end{aligned}$$

which is the same as $D^{(i)}D^{(j)}$ applied to (2), and thereby assumed correct (*use the product rule* twice and recall that $D^{(i)}D^{(j)}$ applied to a single determinant results in zero).

Induction

Finally, *terms proportional to* $a_{n+1,n+1}a_{i,n+1}$ yield

$$
\begin{aligned}
&- D^{(i)} \left|\mathbb{A}\right| \cdot \left|\mathbb{A}_{1,2|1,2}\right| - \left|\mathbb{A}\right| \cdot D^{(i)} \left|\mathbb{A}_{1,2|1,2}\right| \\
&\quad = - D^{(i)} \left|\mathbb{A}_{1|1}\right| \cdot \left|\mathbb{A}_{2|2}\right| - \left|\mathbb{A}_{1|1}\right| \cdot D^{(i)} \left|\mathbb{A}_{2|2}\right| \\
&\qquad + D^{(i)} \left|\mathbb{A}_{1|2}\right| \cdot \left|\mathbb{A}_{2|1}\right| + \left|\mathbb{A}_{1|2}\right| \cdot D^{(i)} \left|\mathbb{A}_{2|1}\right|
\end{aligned}
$$

which is the same as $-D^{(i)}$ applied to (2).

Induction

We have thus been able to cancel out all terms — the extended identity is thus verified.

Thank you for your attention.